# Noise-induced instability: An approach based on higher-order moments

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Noise-induced transitions in the organization of systems far from equilibrium have been of vital interest. Although the effects of additive and multiplicative noise have been widely studied, it is only the multiplicative noise that can be dealt with within the scope of a linear analysis of first moments of the spatiotemporal perturbations, by the application of Novikov's theorem. For the case of additive noise, the corresponding straightforward linear analysis of the first moment throws no light on the effect of the noise on stability conditions. We propose here a simple approach based on higher-order moments to show how additive noise can give rise to noise-induced instability in spatially extended systems, at times leading to pattern formation. Our theoretical analysis is corroborated by numerical simulations on two simple one-component reaction-diffusion systems in two dimensions.

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## I. INTRODUCTION

Noise-assisted and noise-induced processes in nonequilibrium systems have received wide attention over the last two decades. The examples include stochastic resonance, coherence resonance, noise-induced phase transitions, noise-induced transport in Brownian motors, etc., only to name a few [1–4]. Of these, noise-induced instability leading to inhomogeneity and spatial structure formation has proved to be of importance for the study of self-replicating systems, electrodynamic convection in nematic liquid crystals (in a thermal environment with stochastic voltage fluctuations), domain coarsening of patterns, etc. [5–7]. While multiplicative noise processes have a very special role in these and related nonequilibrium phenomena, the role of additive noise is also very subtle. For example, the presence of additive noise is a condition for spatial pattern formation in the self-replicating Gray-Scott model [5], the Swift-Hohenberg model for turbulence [8], etc.

The effect of multiplicative noise can be taken into account in the stability analysis by calculating the first moments of the spatiotemporal perturbations around the steady state, obeying linearized equations, with the help of Novikov's theorem [9]. A systematic development in this spirit was carried out earlier [10] for studying bifurcations in spatially extended systems which exhibit symmetry-breaking instability, and for the determination of the thresholds of the full nonlinear system. Special attraction has been focused on Ginzburg-Landau and Swift-Hohenberg equations in which spatially distributed multiplicative noise was treated in simulations in two dimensions, with an objective of estimating the shift of threshold. Since the multiplicative noise gives rise to a drift term, one can incorporate the effect of this noise in a linear analysis. The corresponding straightforward analysis of linearized, averaged dynamics of the spatiotemporal perturbation for additive noise does not reflect the effect of noise. To explore the effect of additive noise, one therefore has resort to correlation or structure functions within linear approximation and the mean field approach [3]. It has been shown [11] that, based on a Langevin–Fokker-Planck hybrid approach, the mean field model can predict that the application of external noise may advance the location of the critical point for strong spatial coupling, and delay it for weak coupling in spatially extended systems. Within the mean field description, the additive noise can significantly shift the boundaries of phase transitions [12] in nonlinear chains induced by multiplicative noise. It can also change the properties of intermittency and stabilize oscillations [13].

Our aim in this paper is to understand the role of higher-order moments in determining the stability conditions for the homogeneous state of spatially extended one-component systems, in the presence of noise and diffusion. We treat both additive and multiplicative noises in the stochastic dynamics and show that the role of cubic nonlinearity is very special in the context of additive noise. Our theoretical analysis is corroborated by numerical simulations on two specific systems.

## II. THE ONE-COMPONENT MODEL AND NOISE-INDUCED INSTABILITY

We consider the simplest form of the noise-induced reaction-diffusion equation for a one-component system but in multiple dimensions, as a stochastic partial differential equation, with a multiplicative and an additive noise term,

$$u_t = f(u) + D\nabla^2 u + \xi(\vec{x},t)u(\vec{x},t) + \eta(\vec{x},t), \tag{2.1}$$

where $u(\vec{x},t)$ is our concerned variable, $f(u)$ specifies the local reaction kinetics in terms of the functional dependence on the variable, and $D$ denotes the diffusion coefficient. The additive noise term $\eta(\vec{x},t)$ and the multiplicative noise $\xi(\vec{x},t)$ are mutually independent in time and space and are both considered to be Gaussian, white noise sources with zero mean and correlation as given by

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle = 2D_I\delta(\vec{x}-\vec{x}')\delta(t-t'), \tag{2.2}$$

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$$\langle \xi(\vec{x},t)\xi(\vec{x}',t')\rangle = 2D_E\delta(\vec{x}-\vec{x}')\delta(t-t'). \tag{2.3}$$

Here $D_I$ and $D_E$ refer to the strength of additive and multiplicative noises, respectively.

We assume the existence of a spatially uniform steady state $u=u_0$ of the dynamical system such that

$$f(u_0) = 0 \tag{2.4}$$

and furthermore assume that this state is linearly stable with respect to spatially homogeneous perturbation, i.e., stable in the absence of diffusion:

$$\left[\frac{\delta f}{\delta u}\right]_{u=u_0} = f_u < 0. \tag{2.5}$$

Expanding $f(u)$ in a Taylor series about this steady value $u_0$ to all orders in $\delta u$, we have,

$$\frac{\delta(u_0+\delta u)}{\delta t} = f(u_0+\delta u) + D\nabla^2(u_0+\delta u) + \xi(u_0+\delta u) + \eta. \tag{2.6}$$

We now express the spatiotemporal perturbation as

$$\delta u(\vec{x},t) = \overline{\delta u}(\vec{x},t)\cos\vec{k}\cdot\vec{x} \tag{2.7}$$

and additive noise as

$$\eta(\vec{x},t) = \overline{\eta}(\vec{x},t)\cos\vec{k}\cdot\vec{x}. \tag{2.8}$$

Now, we note that $\cos^{2n}x=(1/2^{2n})\left[\sum_{k=0}^{n-1}2\binom{2n}{k}\cos 2(n-k)x+\binom{2n}{n}\right]$ and $\cos^{2n-1}x=(1/2^{2n-2})\left[\sum_{k=0}^{n-1}2\binom{2n-1}{k}\cos(2n-2k-1)x\right]$. By comparing the coefficients of $\cos\vec{k}\cdot\vec{x}$, we have for the envelope function of short wavelength $\nabla^2\overline{\delta u}\ll k^2\overline{\delta u}$

$$\begin{aligned}\frac{\delta[\overline{\delta u}(\vec{x},t)]}{\delta t} &= [f_u - Dk^2]\overline{\delta u} + \frac{1}{3!}f_{u^3}\overline{\delta u}^3\frac{3}{4} + \frac{1}{5!}f_{u^5}\overline{\delta u}^5\frac{5}{8} + \cdots \\ &\quad + \frac{1}{(2n-1)!}f_{u^{2n-1}}\overline{\delta u}^{2n-1}\frac{1}{2^{2n-2}}\frac{(2n-1)!}{(n-1)!\,n!} \\ &\quad + \xi(\vec{x},t)\overline{\delta u} + \overline{\eta}(\vec{x},t)\end{aligned} \tag{2.9}$$

where $f_u$, $f_{u^3}$,..., etc. are the coefficients of the Taylor series of $f(u)$, evaluated at the steady state.

In a discrete lattice of $N$ cells, the stochastic partial differential equation now takes the form (for convenience, we drop the overbar from $\overline{\delta u}$ and $\overline{\eta}$)

$$\begin{aligned}\frac{\delta[\delta u_i(t)]}{\delta t} &= [f_u - Dk^2]\delta u_i + \frac{1}{3!}f_{u^3}\delta u_i^3\frac{3}{4} + \frac{1}{5!}f_{u^5}\delta u_i^5\frac{5}{8} + \cdots \\ &\quad + \frac{1}{(2n-1)!}f_{u^{2n-1}}\delta u_i^{2n-1}\frac{1}{2^{2n-2}}\frac{(2n-1)!}{(n-1)!\,n!} \\ &\quad + \xi_i(t)\delta u_i + \eta_i(t).\end{aligned} \tag{2.10}$$

In this discretized space, the noise correlations acquire [3] the following form:

$$\langle \eta_i(t)\eta_j(t')\rangle = 2C_{i-j}^I\delta(t-t'), \tag{2.11}$$

$$\langle \xi_i(t)\xi_j(t')\rangle = 2C_{i-j}^E\delta(t-t'), \tag{2.12}$$

where $D_I=C_{i-j}^I(\Delta x)^n$ and $D_E=C_{i-j}^E(\Delta x)^n$ for an $n$-dimensional space. We have absorbed the numerical factor for averaging over $\cos^2\vec{k}\cdot\vec{x}$ in $C_{i-j}^I$ and $C_{i-j}^E$.

We now construct the equations of motion for moments with the help of Eq. (2.10):

$$\begin{aligned}\frac{\delta\langle[\delta u_i]\rangle}{\delta t} &= [f_u - Dk^2 + C_E]\langle\delta u_i\rangle + \frac{1}{8}f_{u^3}\langle\delta u_i^3\rangle + \frac{5}{5!\,8}f_{u^5}\langle\delta u_i^5\rangle \\ &\quad + \cdots + \frac{1}{2^{2n-2}(n-1)!\,n!}f_{u^{2n-1}}\langle\delta u_i^{2n-1}\rangle,\end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{\delta\langle[\delta u_i]^2\rangle}{\delta t} &= 2C_I\langle\delta u_i\rangle + 2[f_u - Dk^2]\langle\delta u_i^2\rangle + 2C_E\langle\delta u_i^3\rangle \\ &\quad + \frac{2}{8}f_{u^3}\langle\delta u_i^4\rangle + \cdots \\ &\quad + \frac{2}{2^{2n-4}(n-2)!\,(n-1)!}f_{u^{2n-3}}\langle\delta u_i^{2n-2}\rangle,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\frac{\delta\langle[\delta u_i]^3\rangle}{\delta t} &= 3[f_u - Dk^2 + 2C_I]\langle\delta u_i^3\rangle + \left[\frac{3}{8}f_{u^3} + 9C_E\right]\langle\delta u_i^5\rangle \\ &\quad + \cdots + \frac{3}{2^{2n-4}(n-2)!\,(n-1)!}f_{u^{2n-3}}\langle\delta u_i^{2n-1}\rangle,\end{aligned} \tag{2.15}$$

and so on.

The infinite hierarchy of the above set of equations must be truncated to obtain a closed set of equations for the moments. For this we have

$$\frac{\delta\langle[\delta u_i]^{2n-1}\rangle}{\delta t} = (2n-1)[f_u - Dk^2]\langle\delta u_i^{2n-1}\rangle. \tag{2.16}$$

In deriving the above set of equations (2.13)–(2.16), we have made use of Novikov's theorem [14] in discretized space for Gaussian noise processes [3], viz.,

$$\langle g(\delta u_i)\eta_i\rangle = C_I\langle g(\delta u_i)g'(\delta u_i)\rangle, \tag{2.17}$$

$$\langle g(\delta u_i)\xi_i\rangle = C_E\langle g(\delta u_i)g'(\delta u_i)\rangle. \tag{2.18}$$

For simplification, here we have denoted $C_{i-j}^I$ as $C_I$ and $C_{i-j}^E$ as $C_E$ for $\delta_{ij}$.

The system of equations for the moments (2.13)–(2.16) can be put compactly in the form of a matrix equation as

$$\dot{\vec{L}} = A\vec{L} \tag{2.19}$$

where $\vec{L}$ is a vector comprising the components $\langle[\delta u_i]\rangle$, $\langle[\delta u_i]^2\rangle$, $\langle[\delta u_i]^3\rangle,\ldots,\langle[\delta u_i]^{2n-1}\rangle$; and $A$ is a $(2n-1)\times(2n-1)$ matrix, as shown in the Appendix.

While constructing the set of equations (2.19) from the basic equation (2.10), a close look reveals that noise appearing in all the equations, viz., for

$$[\dot{\delta u_i^2}],[\dot{\delta u_i^3}],[\dot{\delta u_i^4}],\ldots,[\dot{\delta u_i^{2n-1}}]$$

, is multiplicative in nature, although in the parent equation (2.10), for $[\dot{\delta u_i}]$, the noise appears in both multiplicative and additive form. Subscripts $I$ and $E$ in $C_I$ and $C_E$ indicate the parentage of the noise strengths expressed in Eqs. (2.11) and (2.12).

To examine the stability, we now consider the solutions [15] of equations of the moments in the form $\vec{L}(t) \sim e^{\lambda t}$ ($\lambda$ being the frequency) to write the following determinantal equation for the eigenvalue problem:

$$|A - \lambda I| = 0, \tag{2.20}$$

where $I$ is the unit matrix of dimension $(2n-1) \times (2n-1)$.

Expanding the above determinant, we have

$$[f_u - Dk^2 + C_E - \lambda][2(f_u - Dk^2) - \lambda][3(f_u - Dk^2 + 2C_I) - \lambda] \times [4(f_u - Dk^2) - \lambda] \times [5(f_u - Dk^2) - \lambda] \times \cdots \times [(2n-1)(f_u - Dk^2) - \lambda] = 0, \tag{2.21}$$

which can be solved to obtain the eigenvalues

$$\lambda_1 = f_u - Dk^2 + C_E,$$

$$\lambda_2 = 2(f_u - Dk^2),$$

$$\lambda_3 = 3(f_u - Dk^2 + 2C_I),$$

$$\lambda_4 = 4(f_u - Dk^2),$$

$$\lambda_5 = 5(f_u - Dk^2),$$

$$\vdots$$

$$\lambda_{2n-1} = (2n-1)(f_u - Dk^2). \tag{2.22}$$

We are led to the important conclusion that only terms up to the third order contribute toward the noise effect in the time evolution of the variable for a reaction-diffusion system in one dimension.

The condition $k=0$ in the absence of noise corresponds to neglect of diffusion and thus by definition, perturbations of zero wave number are stable when diffusion sets in. Again for $k \neq 0$ and in the absence of noise the system remains homogeneously stable to all orders, since $f_u < 0$.

The instability due to noise sets in [i.e., $\mathrm{Re}\,\lambda(k^2) > 0$] when at least one of the following conditions is satisfied:

$$\text{(i)} \quad C_E > Dk^2 - f_u, \tag{2.23}$$

$$\text{(ii)} \quad C_I > \frac{1}{2}[Dk^2 - f_u]. \tag{2.24}$$

Therefore the range of $k^2$ values for which $\mathrm{Re}\,\lambda$ is positive is $0 < k^2 < (f_u + C_E)/D$ for condition (i) and $0 < k^2 < (f_u + 2C_I)/D$ for condition (ii). The existence of a range of wave numbers indicates that the fluctuation in a certain wave number range only, may exhibit noise-induced instability.

Several pertinent points may now be noted.

(a) In the absence of additive noise, an analysis of the linearized equation for the first moment of $\delta u_i(t)$ for multiplicative noise, using Novikov's theorem, is sufficient to reach the condition (i). However, in the case of additive noise, the analysis of higher-order moments is necessary for the stability analysis to reach the condition (ii).

(b) The existence of a range of wave numbers and the interplay of diffusion and noise are reminiscent of diffusion-driven instability leading to Turing pattern formation [16]. However, while Turing instability requires the diffusion coefficients of the two species to differ significantly and follows from a linear analysis, the present analysis is concerned with moments of arbitrarily high orders. The dispersion relations in the two cases are thus distinctly different.

(c) It is also worth discussing the relation between the condition (2.24) and predictions from the mean field theory about the influence of additive noise. The key point in the later approach is the replacement of the spatial coupling of a particular cell site to its neighbors by coupling to an average value or mean field. Since the average is taken with a probability distribution function satisfying a Fokker-Planck equation, the mean field theory is a Fokker-Planck–Langevin hybrid approach. The method takes care of nonlinearity but discards the details of correlations, to obtain the steady state solutions for mean values and hence determine the stability. The present approach on the other hand, takes care of higher-order moments, which effectively determine the stability of the mean of the spatiotemporal perturbations.

(d) A close look at Eq. (2.9) reveals that the spatiotemporal perturbation with cosine spatial periodicity picks up the harmonic components from cubic and higher-order odd nonlinearities. Again, from the set of equations (2.22) it is clear that cubic anharmonicity plays a lead role in inducing noise-driven instability and spatial inhomogeneity for additive noise.

In what follows, we examine in the next section two simple examples dominated by cubic nonlinearity. For initiation of stationary pattern formation we also explore the role of long-range diffusion in one of these models.

## III. SPECIFIC EXAMPLES AND NUMERICAL SIMULATIONS

### A. The arsenous acid–iodate system

The arsenous acid–iodate model [17,18] is actually a composite of two reactions, viz., the Dushman reaction and the Roebuck reaction. This model has received attention over the last three decades for the study of wave-front propagation. In later years, the model was simplified into a one-variable system [17]. Here, we carry out a study of the one-variable system in the presence of additive noise in a two-dimensional space. The chemical reaction under consideration is [18]

$$IO_3^- + 3H_3AsO_3 + 5I^- \rightarrow 6I^- + 3H_3AsO_4.$$

The rate equation for this reaction has been deciphered to be [19]

$$\frac{d[\mathrm{I}^-]}{dt} = (k_a + k_b[\mathrm{I}^-])[\mathrm{I}^-]([\mathrm{IO}_3^-]_0 - [\mathrm{I}^-])[\mathrm{H}^+]^2. \quad (3.1)$$

With $[\mathrm{I}^-]=u$, our variable of interest, $[\mathrm{H}^+]=h$, a constant, and $[\mathrm{IO}_3^-]_0=I_0$, the initial arsenous acid concentration, a constant, $k_a=k_1$ and $k_b=k_2$, the kinetic constants, the rate equation now takes the form

$$\frac{\delta u}{\delta t} = (k_1 + k_2 u)u(I_0 - u)h^2 = f(u). \quad (3.2)$$

The above equation admits of three steady states of the dynamics, viz.,

$$u_0 = I_0, -\frac{k_1}{k_2}, 0. \quad (3.3)$$

Adding the diffusion terms and an additive noise term, we have

$$\frac{\delta u}{\delta t} = k_1 I_0 h^2 u - k_1 h^2 u^2 + k_2 I_0 h^2 u^2 - k_2 h^2 u^3 + D\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right) + \eta. \quad (3.4)$$

Proceeding as we did in Sec. II, we have

$$f_u = (k_1 + k_2 u)(I_0 - u)h^2 + k_2 u(I_0 - u)h^2 - (k_1 + k_2 u)uh^2. \quad (3.5)$$

The experimentally admissible parameters [19] are given by $k_1=4.5\times10^3 M^{-3}\ \mathrm{s}^{-1}$; $k_2=1.0\times10^8 M^{-4}\ \mathrm{s}^{-1}$, $D=2\times10^{-5}\ \mathrm{cm}^2\ \mathrm{s}^{-1}$, $I_0=5.0\times10^{-3}M$, and $h=7.1\times10^{-3}M$.

At the steady state (say, for $u_0=I_0$), we have in our earlier notation $f_u=-(k_1+k_2 I_0)I_0 h^2$, a negative quantity, indicating a homogeneous stable state. The dispersion relation, i.e., the variation of $\lambda$ vs $k^2$, is given by $\lambda_3=3(f_u-Dk^2+2C_I)$, since all other $\lambda_i$'s $(i=1,2,4,5,\ldots)$ are negative, and $C_E=0$, the multiplicative noise being absent. For the given set of parameters the instability sets in for the noise strength $C_I>0.0635$.

In order to examine the stability analysis numerically we now use the explicit Euler method for the integration of the stochastic partial differential equation (3.4), following the discretization of space and time. A finite system size of $128\times128$ points, with periodic boundary conditions has been chosen. A time interval $\Delta t=0.005$ and a cell size $\Delta x=1.0$ have been found to be appropriate for the purpose. The noise has been discretized and simulated by means of a random Gaussian noise generator. We have carried out our numerical simulations for different values of $C_I$, ranging from 0.0 to 20.0. It has been observed that in the above parameter range, there is a significant change of state from homogeneity [Fig. 1(c)], to an inhomogeneous state with a notable spatial pattern [Fig. 1(d)]. A comparison between the dispersion relations in terms of the plots of $\lambda_3$ vs $k^2$ [Figs. 1(a) and 1(b)] and the numerical simulation for $C_I=0.0$ and $C_I=0.0645$ [Figs. 1(c) and 1(d)] indicates that these results agree very well with the predictions made by our analysis.

### B. The Swift-Hohenberg equation

The Swift-Hohenberg system is a nonlinear model of the reaction-diffusion system used to study the Rayleigh-Benard convection [7,8,20–22]. The model characterized by a cubic nonlinearity and a higher-order diffusion term (the Swift-Hohenberg coupling term) [23], written in terms of the component $u(r,t)$ in two dimensions, is

$$\frac{\delta u(r,t)}{\delta t} = \Gamma u - (1+\nabla^2)^2 u - u^3 \quad (3.6)$$

where $\Gamma$ is a control parameter. Numerical studies of the system have been carried out by a number of groups. García-Ojalvo *et al.* [8] have shown numerically simulated pattern formation in the presence of multiplicative and additive noise terms and also a linear stability analysis in the presence of multiplicative noise. While the earlier analysis is based on structure functions within linear approximation, here we provide an explicit stability analysis for a system having both types of noises in terms of higher-order moments and draw a comparison to the numerical simulations in the various parameter ranges,

$$\frac{\delta u(r,t)}{\delta t} = [\Gamma + \xi(r,t)]u - (1+\nabla^2)^2 u - u^3 + \eta(r,t) \quad (3.7)$$

where $\xi(r,t)$ is a multiplicative noise and $\eta(r,t)$ refers to additive noise. Both of the noise processes are Gaussian and white. The fluctuations of additive noise are of internal origin. On the other hand, the multiplicative noise describes the external fluctuations in the temperature gradient. The primary aim of the model as well as the introduction of noise is to understand the nature of shift of the boundary of instability, which, for example, explains the transition of the system from a homogeneous state to a convective ordered state. Proceeding as earlier, we get the discretized equations as

$$\frac{\delta(\delta u_i)}{\delta t} = [\Gamma + \xi - 1 + 2k^2 - k^4 - 3u_0^2](\delta u_i) - \frac{3}{4}(\delta u_i)^3 + \eta_i, \quad (3.8)$$

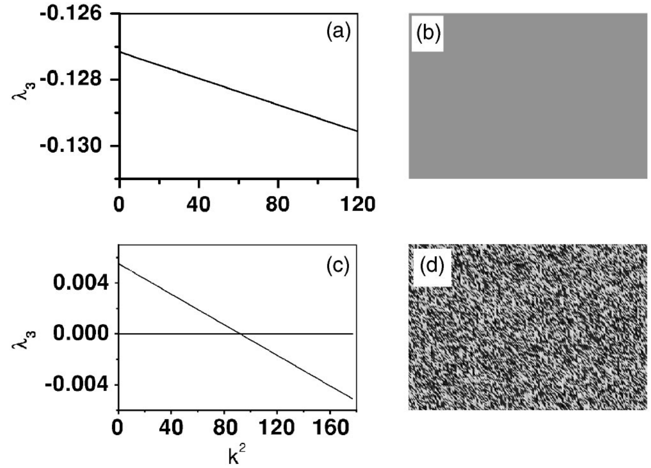


FIG. 1. The arsenous acid–iodate system. (a) Plot of $\lambda_3$ vs $k^2$ ($\lambda_i<0$, for $i\neq3$); (b) time evolved spatial patterns for the parameter set mentioned in the text and $C_I=0.0$, $C_E=0.0$; (c) plot of $\lambda_3$ vs $k^2$ ($\lambda_i<0$, for $i\neq3$); (d) time evolved spatial patterns for the parameter set mentioned in the text and $C_I=0.0645$, $C_E=0.0$.

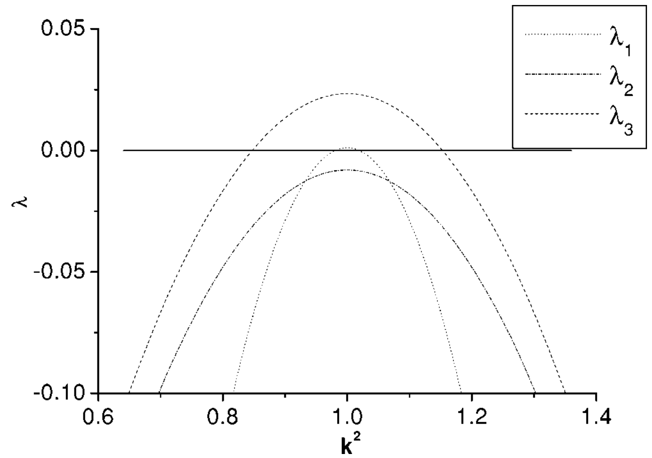


FIG. 2. The Swift-Hohenberg model. Plot of $\lambda_1, \lambda_2$, and $\lambda_3$ vs $k^2$ for dimensionless parameters $\Gamma = 0.37$, $D_E = 0.01$, $D_I = 0.001$.

$$\frac{\delta(\delta u_i)^2}{\delta t} = 2[\Gamma + \xi - 1 + 2k^2 - k^4 - 3u_0^2](\delta u_i)^2 - \frac{3}{2}(\delta u_i)^4 + 2(\delta u_i)\eta_i, \tag{3.9}$$

$$\frac{\delta(\delta u_i)^3}{\delta t} = 3[\Gamma + \xi - 1 + 2k^2 - k^4 - 3u_0^2](\delta u_i)^3 - \frac{9}{4}(\delta u_i)^5 + 3(\delta u_i)^2\eta_i. \tag{3.10}$$

Here, $u_0$ is the steady state value of the concentration.

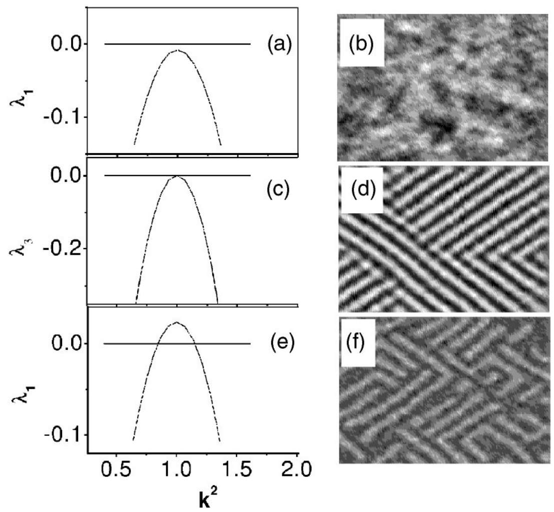


FIG. 3. The Swift-Hohenberg model. (a) Plot of $\lambda_1$ vs $k^2(\lambda_1 = \frac{1}{2}\lambda_2 = \frac{1}{3}\lambda_3 < 0)$; (b) time evolved spatial patterns for dimensionless parameters $\Gamma = 0.37$, $D_E = 0.0$, $D_I = 0.0$; (c) plot of $\lambda_3$ vs $k^2(\lambda_1, \lambda_2 < 0)$; (d) time evolved spatial patterns for dimensionless parameters $\Gamma = 0.37, D_E = 0.0, D_I = 0.001$; (e) plot of $\lambda_1$ vs $k^2(\lambda_2 < 0; 0 < \lambda_3 < \lambda_1)$; (f) time evolved spatial patterns for dimensionless parameters $\Gamma = 0.37$, $D_E = 0.01$, $D_1 = 0.001$.

Averaging over the equations and considering solutions of the moments, we have the following determinantal equation for the eigenvalue problem:

$$\begin{vmatrix} [(\Gamma - 1 + 2k^2 - k^4 - 3u_0^2 + C_E) - \lambda] & 0 & -\frac{3}{4} \\ 2C_I & [2(\Gamma - 1 + 2k^2 - k^4 - 3u_0^2) - \lambda] & 2C_E \\ 0 & 0 & [3(\Gamma - 1 + 2k^2 - k^4 - 3u_0^2 + 2C_I) - \lambda] \end{vmatrix} = 0. \tag{3.11}$$

It is to be noted that we consider only up to the third moment, as the higher terms do not contribute toward the noise effect, as elaborated in Sec. II. The eigenvalues are given by

$$\lambda_1 = (\Gamma - 1 + 2k^2 - k^4 - 3u_0^2 + C_E), \tag{3.12}$$

$$\lambda_2 = 2(\Gamma - 1 + 2k^2 - k^4 - 3u_0^2), \tag{3.13}$$

$$\lambda_3 = 3(\Gamma - 1 + 2k^2 - k^4 - 3u_0^2 + 2C_I). \tag{3.14}$$

Now, for the real steady states $u_0(0, \pm(\Gamma - 1)^{1/2})$ we must have $\Gamma \geqslant 1.0$. We choose the homogeneous steady state $u_0 = 0$ for the present analysis. Proceeding according to our earlier stability analysis, we plot the dispersion relations as the variation of eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ vs $k^2$ in Fig. 2 for $\Gamma = 0.37$, $C_E = 0.042$, and $C_I = 0.0042$. The range of $k^2$ values for which $\lambda(k^2)$ is positive is shown. The largest eigenvalues over a range of admissible wave numbers for different parameter values have been depicted in Figs. 3 and 4 in the presence and absence of additive and multiplicative noises, since the instability is ensured if the largest eigenvalue is positive.

We now carry out numerical simulations on Eq. (3.7) using the explicit Euler method, with a system grid size of $128 \times 128$ cells and with periodic boundary conditions. A time interval $\Delta t = 1.7 \times 10^{-3}$ and a mesh size, $\Delta x = 0.4870$ were taken. The noises have been generated by means of a random Gaussian noise generator. The multiplicative noise has been used in the Stratonovich sense. The value of noise

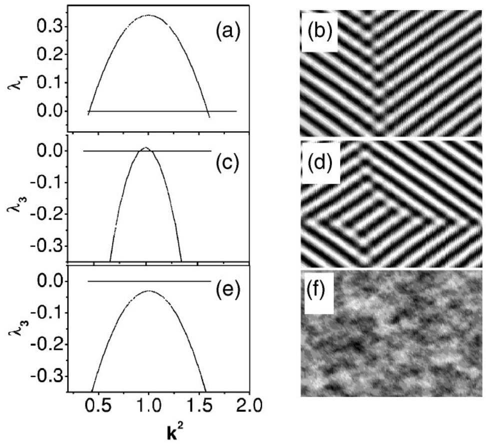


FIG. 4. The Swift-Hohenberg model. (a) Plot of $\lambda_1$ vs $k^2(\lambda_2,\lambda_3<0)$; (b) time evolved spatial patterns for dimensionless parameters $\Gamma=-0.05, D_E=0.1, D_I=0.001$; (c) plot of $\lambda_3$ vs $k^2(\lambda_1,\lambda_2<0)$; (d) time evolved spatial patterns for dimensionless parameters $\Gamma=0.05, D_E=0.0, D_I=0.001$; (e) plot of $\lambda_3$ vs $k^2(\lambda_1,\lambda_2<0)$; (f) time evolved spatial patterns for dimensionless parameters $\Gamma=0.0,\ D_E=0.0,\ D_I=0.001$.

that has been applied is $D_E=C_E(\Delta x)^2$ and $D_I=C_I(\Delta x)^2$.

We choose different values of $\Gamma$, ranging from 0.37 to −0.37, and with $D_E$ and $D_I$ varying between 0.0 and 0.1. It has been observed that for $\Gamma=0.37$, there is a significant change of state from homogeneity [for $D_I=0.0$; Fig. 3(b)] to an inhomogeneous state with a striped pattern [for $D_I=0.001$; Fig. 3(d)]. Again for the multiplicative noise with strength $D_E=0.01$, the presence of additive noise with a finite positive strength gives rise to a labyrinthine pattern [Fig. 3(f)]. A comment on Fig. 3(f) may be pertinent. We find that this pattern is less developed than the one in Fig. 3(d), although the eigenvalue graph in Fig. 3(e) is well inside the instability region. A plausible reason is that the application of both additive and multiplicative noises makes more than one eigenvalue positive $(\lambda_1,\lambda_3>0,\lambda_2<0)$, which may result in the interference of different noise-induced modes, thus leading to a blurring of patterns. The numerically simulated patterns obtained for $\Gamma=0.05$, 0.0, and −0.05, and the corresponding plots of dispersion relations are shown in Fig. 4. The results of the numerical simulations show, in general, a good conformation to the theoretical analysis.

## IV. CONCLUSION

In this paper we have studied the effect of noise in inducing instability on a homogeneous stable state of one-component reaction-diffusion systems. It has been shown that, while it is sufficient to consider the linear equation of motion for the first moment of the spatiotemporal perturbation for multiplicative noise, the analysis of higher-order moments is imperative in the case of additive noise for an understanding of the role of noise.

We now summarize the main conclusions of this study.

(i) We have determined an analytical threshold condition for the instability induced by additive noise on reaction-diffusion systems. Our analysis is equipped to deal with multiplicative noise processes as well. The theoretical estimates of the threshold in model systems correspond fairly well to the results of numerical simulation studies.

(ii) In contrast to mean field approaches, where the details of correlations are neglected, the instability conditions derived by the present approach are based on higher-order moments where coupling to lower orders is appropriately taken care of.

(iii) Our study indicates that the cubic and odd nonlinearities are of special type since their contribution is picked up as the harmonic components that determine the structure of stability matrix and dispersion relations. Since cubic nonlinearity is almost ubiquitous in all major areas of nonlinear dynamics, we believe that the role played by this nonlinearity in the context of additive noise, in inducing instability and pattern formation, is generic and makes the perspective of noise-induced transition much wider.

We hope this approach will be useful for exploring noise in pattern formation and selection in spatially extended systems with more than one variable.

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## APPENDIX

The $(2n-1)$ component vector $\vec{L}$ and the matrix $A$ of dimension $(2n-1)\times(2n-1)$ are given by

$$\vec{L}=\begin{pmatrix}\langle[\delta u_i]\rangle\\ \langle[\delta u_i]^2\rangle\\ \langle[\delta u_i]^3\rangle\\ \langle[\delta u_i]^4\rangle\\ \langle[\delta u_i]^5\rangle\\ \cdot\\ \cdot\\ \cdot\\ \langle[\delta u_i]^{2n-1}\rangle\end{pmatrix} \quad (A1)$$

and

$$
A = \begin{pmatrix}
[f_u - Dk^2 + C_E] & 0 & \frac{1}{8}f_{u^3} & 0 & \frac{1}{4\,!\,8}f_{u^5} & \dots & 0 & \left[\frac{1}{2^{2n-2}(n-1)!}\frac{1}{n!}f_{u^{2n-1}}\right] \\
2C_I & 2(f_u - Dk^2) & 2C_E & \frac{2}{8}f_{u^2} & 0 & \dots & \left[\frac{2}{2^{2n-4}(n-2)!}\frac{1}{(n-1)!}f_{u^{2n-3}}\right] & 0 \\
0 & 0 & 3(f_u - Dk^2 + 2C_I) & 0 & \left[\frac{3}{8}f_{u^3} + 9C_E\right] & \dots & 0 & \left[\frac{3}{2^{2n-4}(n-2)!}\frac{1}{(n-1)!}f_{u^{2n-3}}\right] \\
0 & 0 & 0 & 4(f_u - Dk^2) & 12C_I & \dots & \left[\frac{4}{2^{2n-6}(n-3)!}\frac{1}{(n-2)!}f_{u^{2n-5}}\right] & 0 \\
0 & 0 & 0 & 0 & 5(f_u - Dk^2) & \dots & 0 & \left[\frac{5}{2^{2n-6}(n-3)!}\frac{1}{(n-2)!}f_{u^{2n-5}}\right] \\
 & & & & \vdots & & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & (2n-1)(f_u - Dk^2)
\end{pmatrix}. \tag{A2}
$$


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